Solving Equations Involving the Catenary with MLAB

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A 10 foot rope hangs in the form of a catenary with one end attached to a pier at a point 3 feet above the water's surface, and the other end attached at the same height to a boat. If the lowest point on the catenary just touches the water, how far is the boat from the pier? Another question is: if the distance from the boat to the pier is fixed, does the lowest point on the catenary submerge or clear the surface of the water as the tide ebbs? as the tide rises?
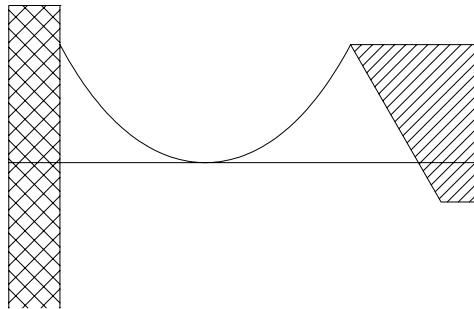

Figure 1: A rope fixed to a pier and a boat. The lowest point on the suspended rope touches the water's surface.

It has been known since Leibnitz and Huygens' time, circa 1691, that a rope or chain tethered at each end hangs in a vertical plane in the shape of a catenary curve. Symon[1971] gives the mathematical equation of a catenary curve as:

$$y(x) = \beta + \gamma \cosh(\frac{x}{\gamma} + \alpha)$$

where $\cosh(t)$ is the hyperbolic cosine function, $\cosh(t) = \frac{e^t + e^{-t}}{2}$.

To describe our situation, the parameters α, β, and γ must be chosen so that the catenary passes through both the fixed fastening point on the pier and the vertically-changeable point representing the bow of the boat, and such that the length of the catenary has the fixed value 10.

The length of the catenary with abscissae (x-coordinates) in the interval $[x_0, x_1]$ is given by:

$$L(x_0, x_1) = \gamma \left[\sinh(\frac{x_1}{\gamma} + \alpha) - \sinh(\frac{x_0}{\gamma} + \alpha) \right]$$

where $\sinh(t)$ is the hyperbolic sine function, $\sinh(t) = \frac{e^t - e^{-t}}{2}$. The length of the catenary curve satisfies this equation because the length of any differentiable curve $y(t)$ between the points $(t_0, y(t_0))$ and $(t_1, y(t_1))$ is:

$$\int_{t_0}^{t_1} (1 + (\frac{dy}{dt})^2)^{\frac{1}{2}} dt$$

We choose a coordinate system in which the origin is at the point halfway between the two fastening points of our catenary when the water level is 3 feet below the fastening point on the pier. Take $\bar{x}$ to be the distance between the boat and the pier in this situation. The minimum of the catenary is at the point $(0, -3)$, the suspension point on the pier is at $(-\frac{\bar{x}}{2}, 0)$, and the suspension point on the bow of the boat is $(\frac{\bar{x}}{2}, 0)$. The following equations must be satisfied: $y(-\frac{\bar{x}}{2}) = 0$, $y(\frac{\bar{x}}{2}) = 0$, $y(0) = -3$, and $L(-\frac{\bar{x}}{2}, \frac{\bar{x}}{2}) = 10$. Then we have:

$$
\begin{aligned}
y(0) = -3 &= \beta + \gamma \cosh(\alpha) \\
y(\frac{\bar{x}}{2}) = 0 &= \beta + \gamma \cosh(\frac{\bar{x}}{2\gamma} + \alpha) \\
y(-\frac{\bar{x}}{2}) = 0 &= \beta + \gamma \cosh(-\frac{\bar{x}}{2\gamma} + \alpha)
\end{aligned}
$$

$$L(\frac{-\bar{x}}{2}, \frac{\bar{x}}{2}) = 10 \;\; = \;\; \gamma \, [\sinh(\frac{\bar{x}}{2\gamma} + \alpha) - \sinh(-\frac{\bar{x}}{2\gamma} + \alpha)]$$

where $\bar{x}$ is the distance separating the pier and the boat. Here we have four equations with the four unknowns α, β, γ, and $\bar{x}$. We want to solve for the value $\bar{x}$, and this requires determining α, β, and γ as well.

The MLAB mathematical modeling program provides several ways of solving a set of equations such as this. Here we describe two methods: one using the `ROOT` operator and the other using the `MINIMIZE` operator.

To begin, we reduce our four equations in four unknowns to one equation in the unknown variable γ. Setting the expression for $y(\frac{\bar{x}}{2})$ equal to the expression for $y(-\frac{\bar{x}}{2})$ and cancelling common terms, one finds $\cosh(\frac{\bar{x}}{2\gamma} + \alpha) = \cosh(-\frac{\bar{x}}{2\gamma} + \alpha)$. Writing the cosh functions in terms of exponential functions and multiplying the resulting equation by 2, one finds:

$$e^{\frac{\bar{x}}{2\gamma}} \cdot e^{\alpha} + e^{-\frac{\bar{x}}{2\gamma}} \cdot e^{-\alpha} = e^{-\frac{\bar{x}}{2\gamma}} \cdot e^{\alpha} + e^{\frac{\bar{x}}{2\gamma}} \cdot e^{-\alpha}$$

This condition is satisfied if $\alpha = 0$.

Next, substituting $\alpha = 0$ into the equation for $y(0)$ and solving for β, one finds $\beta = -(\gamma + 3)$. Then substituting $\alpha = 0$ and $\beta = -(\gamma + 3)$ in the equation for $L(\frac{-\bar{x}}{2}, \frac{\bar{x}}{2})$ we obtain

$$10 = \gamma[\sinh(\frac{\bar{x}}{2\gamma} + \alpha) - \sinh(-\frac{\bar{x}}{2\gamma} + \alpha)]$$

and solving for $\bar{x}$, one finds $\bar{x} = 2\gamma \sinh^{-1}(\frac{5}{\gamma})$, where $\sinh^{-1}(t)$ is the inverse hyperbolic sine function.

Finally, substituting $\alpha = 0$, $\beta = -(\gamma + 3)$, and $\bar{x} = 2\gamma \sinh^{-1}(\frac{5}{\gamma})$ into the equation for $y(\frac{\bar{x}}{2})$, one obtains the following equation for γ:

$$0 = -3 + \gamma \, [\cosh(\sinh^{-1}(\frac{5}{\gamma}) - 1)].$$

The MLAB `ROOT` operator finds a value of a variable within a specified interval where a given function of that variable equals zero. The form of the `ROOT` operator is `ROOT(X,A,B,E)` where `X` is the name of the unknown variable, `A` and `B` are numbers or expressions that specify the range of values within which the root-value lies, and `E` is a mathematical expression involving the

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unknown variable. Thus `ROOT(X,A,B,E)` = `v` such that `v` is a number in the interval `[A,B]` and `E(v)` = 0. The `ROOT` operator employs a hybrid algorithm that mixes bracketing, secant, Newton, and bisection methods. These methods are described in Press[1992].

The equation given above involving γ can be solved and the other unknowns evaluated, with the following MLAB commands:

```
FCT F() = ROOT(G,0,100,-(3+G)+G*COSH(ASINH(5/G)))
A = 0; C = F(); XB = 2*C*ASINH(5/C); B = -(C+3);
TYPE A,B,C,XB
```

Here, `COSH` and `ASINH` are built-in MLAB functions that evaluate the hyperbolic cosine and inverse hyperbolic sine, respectively. The `TYPE` command prints the following results to the screen:

```
A = 0
B = -5.66666667
C = 2.66666667
XB = 7.39356993
```

Another method of solving the four equations consists of using the MLAB `MINIMIZE` operator to find the values of α, β, γ, and $\bar{x}$ that make a least-squares objective function evaluate to the minimum value of 0. The least-squares objective function in this problem is given by:

$$\text{obj}() = [10 - L(-\frac{\bar{x}}{2}, \frac{\bar{x}}{2})]^2 + [0 - y(\frac{\bar{x}}{2})]^2 + [0 - y(-\frac{\bar{x}}{2})]^2 + [3 + y(0)]^2.$$

Here are the MLAB commands that use the `MINIMIZE` operator to solve the problem:

```
FCT Y(X) = B+C*COSH((X/C)+A)
FCT L(X0,X1) = C*(SINH((X1/C)+A)-SINH((X0/C)+A))
FCT OBJ() = (10-L(-XB/2,XB/2))^2+(Y(XB/2))^2+(Y(-XB/2))^2+(3+Y(0))^2
A = 1; C = 1; XB = 1; B = 1
MINIMIZE(OBJ,XB,A,B,C)
TYPE A,B,C,XB
```

With the first 3 commands, we define `Y(X)` to be the general form of the catenary function, `L(X0,X1)` to be the distance between the endpoints along

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the catenary, and `OBJ()` to be the least-squares objective function for the system of equations. The assignment statements provide initial guesses for the parameters `A`, `B`, `C`, and `XB` that appear in the definitions of the functions. Then the `MINIMIZE` operation does a variable metric method search (as described in Press[1992]) of the $(\alpha,\beta,\gamma,\bar{x})$-parameter space, starting from the point $(1,1,1,1)$. The first argument to `MINIMIZE` is the name of the function for which a minimum is to be found; the remaining arguments are the names of the parameters to be varied. The `MINIMIZE` operator changes the values of the parameters listed in the argument list. The results printed to the screen by the last `TYPE` command are:

```
A = 8.86949601E-10
B = -5.66666669
C = 2.66666669
XB = 7.39356995
```

Both the `ROOT` and `MINIMIZE` methods determined the separation of the pier and the boat to be approximately 7.39 feet.

The MLAB `FIT` and `MINIMIZE` operators can now be used to investigate how the lowest point on the catenary changes as the right endpoint moves down or up vertically, i.e. as the tide lowers or rises. Although the lowest point of the catenary shifts to the right or left of the y-axis as the boat rises or falls with the tide, we continue to use the coordinate system described above in which the y-axis passes through the minimum of the catenary formed when the boat and the pier *are* at the same level. Note that the bow of the boat is always three feet above the water level.

Now suppose the water level changes in the coordinate system by δ feet, so that the boat vertically-rises or falls to place the bow of the boat at $(\frac{\bar{x}}{2}, \delta)$ or $(\frac{\bar{x}}{2}, -\delta)$. δ is constrained to be less than or equal to $\sqrt{100 - (\frac{\bar{x}}{2})^2}$ by the Pythagorean theorem.

Thus we can compute the minimum point on the catenary by determining the catenary curve such that $y(-\frac{\bar{x}}{2}) = 0$, $y(\frac{\bar{x}}{2}) = \delta$, and $L(-\frac{\bar{x}}{2}, \frac{\bar{x}}{2}) = 10$.

```
/* Compute MH to be the maximum height above (below) the fastening
   point on the pier that the boat can attain if the rope is 10 feet
   long and the pier and boat are separated by XB feet. */
MH = SQRT(100-XB^2);
```

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```
/* Compute a vector of 21 equally-spaced height values ranging from
   -MH to MH, excluding MH and -MH */
D = -MH:MH!23
DELETE D ROW (1,23)

/* L(-XB/2,XB/2) = 10, and ML = the corresponding data point */
ML = LIST(-XB/2,XB/2,10)'

/* define linear constraints for XMIN, B, C */
CONSTRAINTS Q1 = {B<0,C>0}
CONSTRAINTS Q2 = {XMIN >= -XB/2, XMIN <= XB/2}

XMIN = 0; /* initial guess at minimum's abscissa */

/* loop over 21 different heights, D[J], J = 1,2,...,21 */
FOR J = 1:21 DO {
    /* MC = 2X2 matrix containing endpoints of the catenary
       when the boat is at (XB/2,D[J]) */
    MC = SHAPE(2,2,LIST(-XB/2,0,XB/2,D[J]));

    /* adjust (A,B,C) to fit the catenary to the endpoints and length */
    FIT (A,B,C), Y TO MC, L TO ML, CONSTRAINTS Q1;

    /* find the minimum point (XMIN,YMIN) on the catenary, leaving
       A,B, and C fixed, constraining (-XB/2)<=XMIN<=(XB/2) */
    YMIN = MINIMIZE(Y,XMIN,Q2);

    /* store the results in the matrix MRES */
    MRES ROW J = LIST(D[J],XMIN,YMIN,A,B,C)';
    }
```

The MLAB FIT operator uses a "tuned" version of the Marquardt-Levenberg
curve-fitting method described in Marquardt[1963] and Smith[1970]. The
quadratic programming algorithm, which the curve-fitting algorithm needs
to converge within linear constraints is described in Shrager[1972].

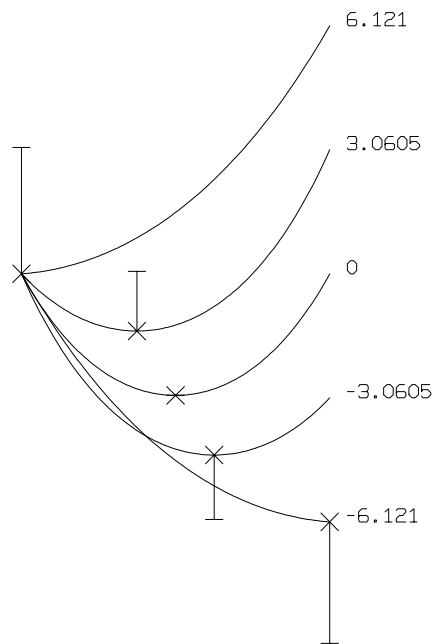Five of the resulting catenaries are graphed via MLAB in Figure 2.

Figure 2: Catenaries for 5 different tide levels. The minimum of each catenary is marked with × and connected to a horizontal dash mark which represents the water level.

The computed minimum of each of the 21 catenaries and the corresponding water levels are shown in Figure 3.
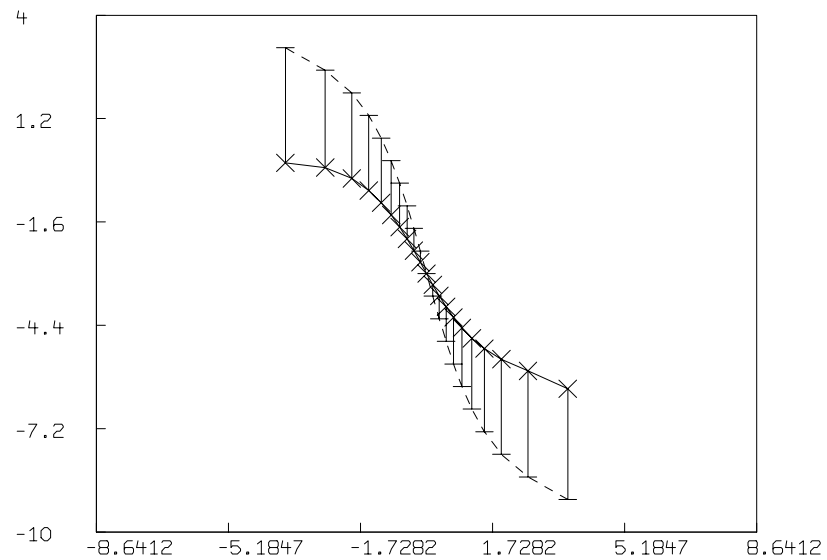
Figure 3: The minima of 21 catenaries marked with × connected to horizontal dash marks which represent water level.

If the tide ebbs, thereby lowering the boat with respect to the pier, the rope clears the water. If the tide rises, thereby raising the boat with respect to the pier, the rope is partially submerged.

This paper has demonstrated the `ROOT`, `MINIMIZE`, and `FIT` operators in MLAB within the context of a catenary problem. MLAB can be used for studying many other problems in physics and engineering.

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