# Linear System Deconvolution

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A one-input linear system is an operator, $H$, which maps an *input* function, $x$, to a corresponding *output* function, $y$. Thus $y(t) = (Hx)(t)$. $H$ is a linear operator, so that $H(ax + bz) = a(Hx) + b(Hz)$.

A *shift-invariant* linear system has the property that $H(x(t - s)) = (Hx)(t-s)$. A shift-invariant linear system, $H$, must be frequency-preserving, so that $H(a \cdot \cos(st + b)) = c \cdot \cos(st + d)$ for some values $c$ and $d$. The admissible input functions, $x$, are just those complex-valued functions which possess a Fourier-Stieljes transform.

Every shift-invariant linear system operator $H$ has an associated complex-valued function $h$, called the *system-weighting* function of $H$, such that $(Hx) = x * h$ where $*$ denotes the *convolution* operator; thus $(x * h)(t) = \int x(r)h(t-r)dr$. Often $h$ is called the *impulse-response* function of $H$, since $h = \delta_0 * h$, where $\delta_0$ is the Dirac $\delta$-function with its spike at 0. Let $Hx = y$. Note that saying $y = x * h$ shows that each value, $y(t)$, is a certain weighted "sum" of the values of $x$, where the value $x(r)$ is weighted by $h(t - r)$.

In order that $y(t)$ depend only on the $x$-values $x(r)$ with $r \le t$, we must have $h(t) = 0$ for $t < 0$. A linear system with such a weighting function is called *physically-realizable*; it corresponds to some real-time processor which can input $x$ and output $y$ in real-time. Such a processor may, of course, involve memory, but not a delay due to "reading ahead".

A linear system operator $H$ is *stable* if $Hx$ is bounded when $x$ is bounded; thus finite input cannot cause the output of a stable system to "blow-up". If $H$ is a stable shift-invariant linear system operator with the impulse-response function $h$, then the Fourier transform of $h$ exists and the complex-valued function $h^\wedge$ is called the *frequency-response* function of $H$, and is such that $(Hx)^\wedge = x^\wedge h^\wedge$. ($\wedge$ denotes the Fourier transform operator and $\vee$ denotes the inverse Fourier transform operator.) Often $h^\wedge$ is called the *transfer* function of $H$.

We may write $h^\wedge$ in polar form as $h^\wedge(s) = |h^\wedge(s)|e^{i\phi(s)}$, where $\phi(s)$ is the phase-shift function of $h$. If $h$ is real then $|h^\wedge(s)| = M(s)/(2 - \delta_{s0})$

where $M(s)$ is the amplitude function of $h$. In any event, $|h^\wedge(s)|$ is called the *gain* function of $H$ and $\phi$ is called the *phase-shift* function of $H$, since if the input $x(t)$ is a complex oscillation $Ae^{i(2\pi st+q)}$, then the output $(Hx)(t)$ is $|h^\wedge(s)|Ae^{i(2\pi st+q+\phi(s))}$, which is just an oscillation of the same frequency, $s$, whose amplitude is multiplied by the gain $|h^\wedge(s)|$ and whose phase is shifted by the phase-shift value $\phi(s)$. This is just a special case of the relation $(Hx)^\wedge = x^\wedge h^\wedge$.

When the system-weighting function, $h$, is real, then the frequency-response function $h^\wedge$ is hermitian, *i.e.* $h_R^\wedge = h^{\wedge *}$, and the gain and phase-shift functions are real and $|h^\wedge(s)|$ is even and $\phi(s)$ is odd. In this case $H$ preserves real signals, *i.e.* $Hx$ is real whenever $x$ is real.

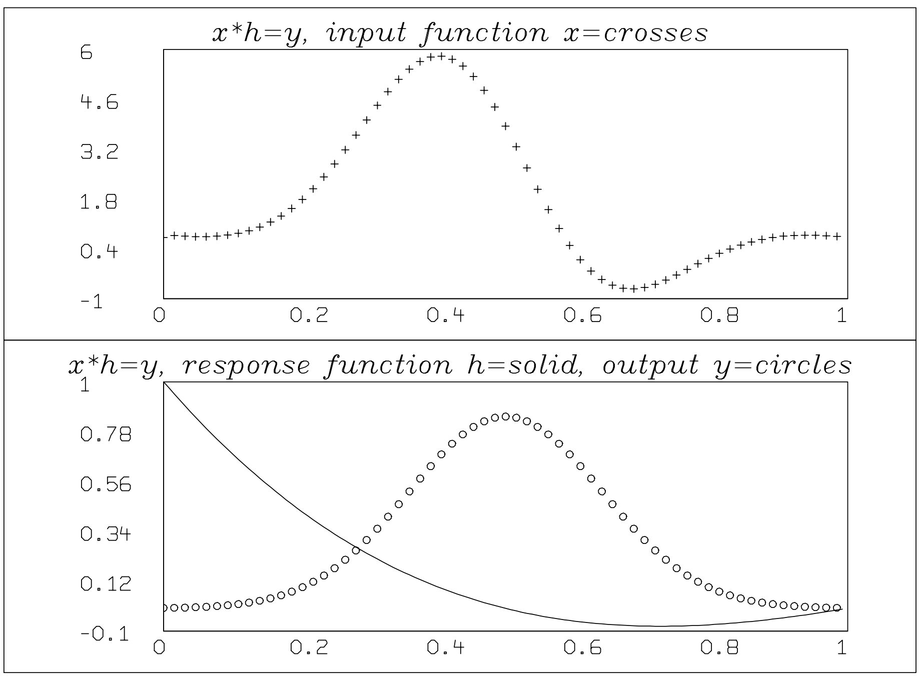
Cascading two stable shift-invariant linear systems $H_1$ and $H_2$ results in a linear system $H_2H_1$ whose output is $(H_2(H_1x))$, and the frequency-response function is $h_1^\wedge h_2^\wedge$, so the gain function is $|h_1^\wedge(s)| \cdot |h_2^\wedge(s)|$ and the phase-shift function is $\phi_1(s) + \phi_2(s)$.

Given the input $x$ and the output $y$ of a stable shift-invariant linear system we may determine the frequency-response function $h^\wedge(s)$ as $y^\wedge(s)/x^\wedge(s)$ at each frequency, $s$, which appears in $x$, *i.e.* for which $x^\wedge(s) \neq 0$. A test input function, $x$, constructed for the purpose of determining $h^\wedge$ should thus have a broad spectrum. In the same way, given the output function $y$ and the frequency-response function $h^\wedge$ (possibly determined by a prior computation based on known input and output), we can obtain the input function as $x = (y^\wedge/h^\wedge)^\vee$.

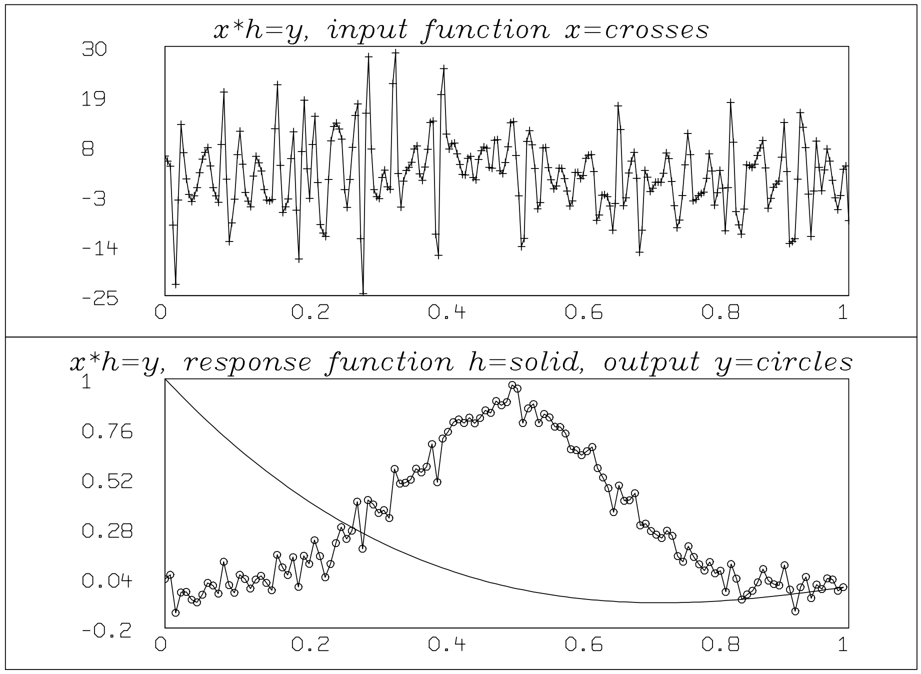
Let us look at an example in MLAB showing how the unknown input function $x$ is determined from a numerically-given impulse-response function $h$ and an observed output function $y$. For computational purposes, $h$ and $y$ are extended periodically; we can can compensate for this treatment by suitably appending zeros to the sample sequences defining $h$ and $y$.

```
h=read(hsamples,128,2)
y=read(ysamples,128,2)

th=dft(h&'0)
ty=dft(y&'0)

tx=cdiv(ty col 2:3, th col 2:3)
tx= (th col 1)&'tx
x= idft(tx) col 1:2

draw h
```

```
draw y row (1:128:2) lt none pt circle ptsize .008
top title "x*h=y, response function h=solid, output y=circles", font 17
frame 0 to 1, 0 to .5
w1=w

draw x  row (1:128:2) lt none pt crosspt ptsize .008
top title "x*h=y, input function x=crosses", font 17
frame 0 to 1, .5 to 1
view
```




Here is an example where the input function contains noise.


x*h=y, input function x=crosses
30
19
8
-3
-14
-25
0
0.2
0.4
0.6
0.8
1
x*h=y, response function h=solid, output y=circles
1
0.76
0.52
0.28
0.04
-0.2
0
0.2
0.4
0.6
0.8
1