File: /General/MLAB-Text/Papers/lrc/lrc.tex

Electric Circuit Dynamics

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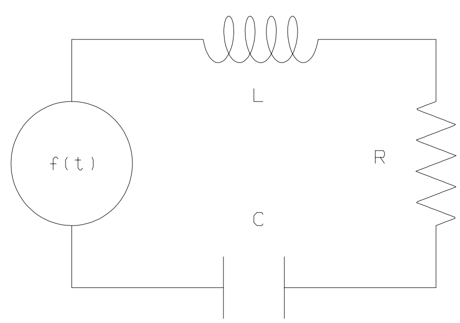
This example demonstrates MLAB's differential equation-solving facilities, the use of MLAB's Fourier transform operations, and MLAB graphics in the context of the classic problem of analyzing an LRC circuit. A circuit containing a coil with an inductance of L henries, a resistor with a resistance of R ohms, and a capacitor with a capacitance of C farads in series is traditionally called an LRC circuit. Consider the LRC circuit shown below which also contains a voltage source component which exhibits a voltage drop of $-f(t)$ volts at time t measured clockwise across the voltage source component. (This picture was constructed using MLAB.)



When the voltage source component is switched into the circuit at time 0, a current will begin to flow. Let $I(t)$ be the current flowing in the circuit at time t, measured in amperes.

By defining what a flowing current consists of, the direction the current flows may be specified arbitrarily, since one direction is the direction of the actual flow of electrons from "hole" to "hole", and then there is a virtual flow of "holes" from electron to electron in the other direction. We shall take a *positive* amount of *charge* (measured in coulombs) to be a deficit of electrons, and we shall take a *negative* amount of *charge* (measured in coulombs) to be an excess of electrons.

We shall take a *positive current* (measured in amperes) to be a flow of positive charge, *i.e.*, "holes", in the established direction of flow, or equivalently, a flow of electrons in the opposite direction, and a *negative current* (measured in amperes) to be a flow of electrons, *i.e.*, a negative amount of charge, in the established direction of flow, or equivalently, a flow of "holes", *i.e.*, a positive amount of charge, in the opposite direction.

We take the voltage source terminal adjacent to the inductor coil in our circuit diagram to be the positive terminal, and the other voltage source terminal adjacent to the capacitor in our circuit diagram to be the negative terminal.

When we have a positive voltage drop, measured as the potential difference between the positive terminal of the voltage source and the negative terminal of the voltage source, the direction of positive current flow is from the positive terminal to the negative terminal of the voltage source, *i.e.*, clockwise in our circuit as drawn. (This is the opposite of the potential difference we shall use later on.) In this case, positive charge (*i.e.*, an amount of "holes") flows from this positive terminal of the voltage source component to first enter the inductor coil, that is, clockwise. The current in our circuit at time t, $I(t)$, will be positive when positive charge is flowing clockwise, and $I(t)$ will be negative when positive charge is flowing counterclockwise.

Let $Q(t)$ be the charge-difference on the capacitor at time t, measured in coulombs. This charge-difference is just the difference between the charges on the first capacitor plate encountered going in the clockwise direction from the positive terminal of the voltage source and on the second capacitor plate encountered going in this direction. The charge is identical throughout the branch of our circuit from the positive terminal of the voltage source to the first capacitor plate, and the charge is separately identical throughout the branch of our circuit from the second capacitor plate to the negative terminal of the voltage source.

We have $dQ(t)/dt = I(t)$.

The voltage drop across the resistor at time t is given by Ohm's law as $R \cdot I(t)$.

The voltage drop across the capacitor at time t is $Q(t)/C$.

The voltage drop across the coil at time t is $L(dI(t)/dt)$.

And, as mentioned above, the voltage drop across the voltage source component is $-f(t)$ at time t.

For each voltage drop across a circuit component, we (conceptually) measure the voltage relative to "ground", v_1, just *before* the circuit component appears in the circuit going in the clockwise direction – *i.e.*, the direction of positive current flow, and we measure the voltage relative to "ground", v_2, just *after* the circuit component terminates in the circuit going in the clockwise direction, and we take the voltage drop to be $v_1 - v_2$. For the voltage source, the voltage drop measured going from the negative terminal to the positive terminal in a clockwise direction in our circuit diagram as drawn is a negative value when our voltage source is a battery with positive and negative terminals oriented as discussed above.

Note if positive current is flowing counterclockwise in our circuit as drawn, our voltage drops for passive circuit components will be negative, and for active circuit components will be positive.

Finally, by Kirchhoff's first law, the sum of the voltage drops across each of the circuit components is 0. Thus, we have

$$L\frac{dI(t)}{dt} + \frac{Q(t)}{C} + RI(t) - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t), \text{ or}$$

$$L\frac{d^2Q(t)}{dt^2} + \frac{Q(t)}{C} + R\frac{dQ(t)}{dt} - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t), \text{ or}$$

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{I(t)}{C} - \frac{df(t)}{dt} = 0.$$

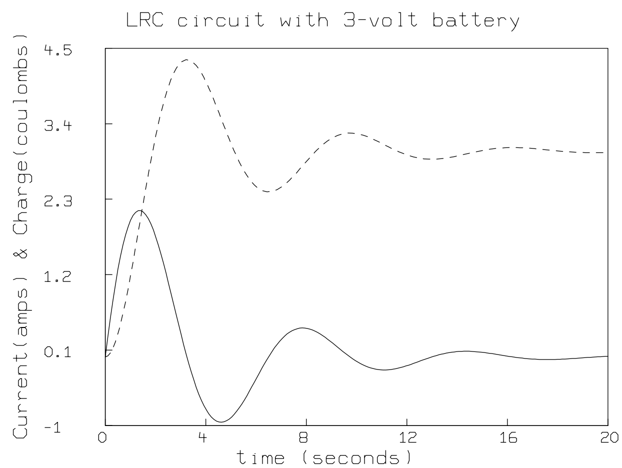
Take $L = 1$ henry, $C = 1$ farad, and $R = .5$ ohms. (Actually, 1 farad is far too large a capacitance to be feasible; for a practical circuit, all these values should be scaled so that we have a capacitance in microfarads. Let us consider the situation where $f(t) = 3$, *i.e.*, our voltage source is a 3-volt battery with its positive terminal appearing just counterclockwise from the induction coil and its negative terminal appearing just clockwise from the capacitor in our circuit diagram. In this case the voltage drop across the battery from negative to positive terminal is $v_1 - v_2 = -3$ volts where v_1 is the voltage due to the difference of charge relative to ground at the negative terminal and v_2 is the voltage due to the difference of charge relative to ground at the positive terminal. To obtain a correct non-zero potential difference, these measurements of v_1 and v_2 to ground must be taken when the battery has chemically transported current flowing through it. A "floating" battery has no measurable potential difference computed by separate measurements between ground and its two terminals since its potential in this "floating" case is the chemical potential represented by the reactions in the battery being held at equilibrium; there is no appreciable charge excess or deficit at the battery terminals.

Suppose further that there are two switches isolating this battery from our (open) circuit. At time 0 both switches are simultaneously thrown and current can flow. (Actually, from the just prior remarks, for a battery, only one switch is required.) The differential equations describing this circuit are:

$$L\frac{dI(t)}{dt} + \frac{Q(t)}{C} + RI(t) - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t)$$

with $f(t) = 3, I(0) = 0$ and $Q(0) = 0$.

The graphs of I and Q for $0 \le t \le 20$ are shown below. The current graph is a solid line, and the charge-difference graph is a dashed line.



The above graph is produced in MLAB with the following commands.

```
/* Define the differential equations governing current flow */
fct i't(t)=(f(t)-r*i -q/c)/l
fct q't(t)=i
fct f(t) = 3
l = 1; c =1; r =.5

/* Provide initial conditions for current flow */
init i(0) = 0
init q(0) = 0

/* Set time-vector  */
tv = 0:20!200

/* Solve the differential equations governing current flow.
   We will have:
    m col 1 = t, m col 2 = I, m col 3 = i't, m col 4 = q, m col 5 = q't */
m = integrate(i't,q't,tv)

/* draw the current flow vs time and charge vs time  */
draw m col (1,2)                          /* I  */
draw m col (1,4)  color green line dashed   /* Q  */
left title "Current(amps) & Charge(coulombs)"
```

```
bottom title "time (seconds)"
top title "LRC circuit with 3-volt battery"
view
```

The steady-state value of I is 0 amperes and the steady-state value of Q is 3 coulombs countering the +3 volt battery. Thus when the capacitor reaches full charge, where the potential difference between the positive terminal of the battery and the first plate of the capacitor connected to this positive terminal is 0 and the potential difference between the negative terminal of the battery and the second plate of the capacitor connected to this negative terminal is also 0, the capacitor now acts as an open switch. The charge difference in the capacitor is 3 coulombs in this steady state because the units of amperes, ohms, volts, and coulombs are defined such that a force of 1 volt drives 1 ampere of current across a resister of 1 ohm and 1 coulomb is the amount of charge carried by a 1 ampere current in 1 second. An accumulation of 1 coulomb of charge on a capacitor plate in excess of the charge on the other capacitor plate corresponds to a voltage difference of 1 volt. Indeed, relative to a neutral ground, we have 1.5 volts of positive charge in the circuit branch between the positive terminal of the battery and the first plate of the capacitor, and −1.5 volts of positive charge in the circuit branch between the second plate of the capacitor and the negative terminal of the battery.

We have a voltage of 3 volts driving a current across a resistor of .5 ohms, so we might expect a maximum current of 6 amperes, diminished by the accumulation of charge on the capacitor, and oscillating due to the opposition of the induction coil. Indeed, we see a maximum current of about 2.2 amperes is obtained. When a charge-difference of 3 coulombs settles in the capacitor, the current in our circuit becomes zero.

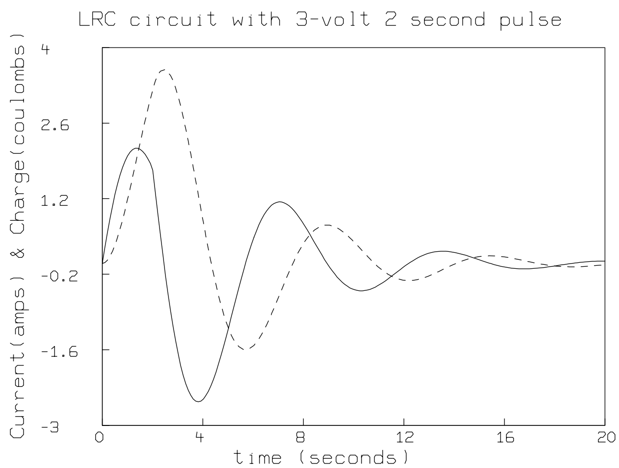
Note the equations $\frac{d^2I(t)}{dt^2} = \left(\frac{df(t)}{dt} - R\frac{dI(t)}{dt} - \frac{I(t)}{C}\right)/L$ and $\frac{dQ(t)}{dt} = I(t)$ with $I(0) = 0$, $Q(0) = 0$, and $\frac{dI(0)}{dt} = \left(f(0) - RI(0) - \frac{Q(0)}{C}\right)/L = f(0)/L$. are equivalent to the equations above with exactly the same solutions.

When the "driving force" $f(t)$ is defined as the "square pulse" $f(t) = \text{if } t < 2 \text{ then } 3 \text{ else } 0$ then we have

$$L\frac{dI(t)}{dt} + \frac{Q(t)}{C} + RI(t) - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t),$$

$$\text{with } f(t) = \text{if } t < 2 \text{ then } 3 \text{ else } 0, I(0) = 0, \text{ and } Q(0) = 0.$$

The graphs of I and Q for $0 \le t \le 20$ with this driving force function are shown below. The current graph is a solid line, and the charge-difference graph is a dashed line.



Again the current and charge-difference in the circuit decays to a steady state, but in this case the current approaches 0 amperes, and the capacitor charge-difference approaches 0 coulombs. The initial energy input in the first 2 seconds is converted to heat by the resistor.

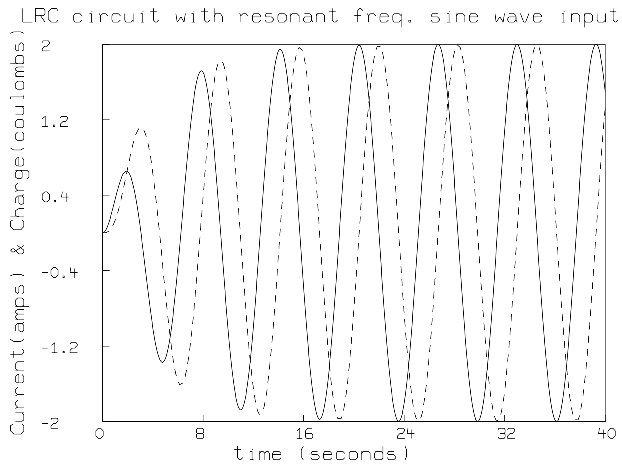
When the "driving force" $f(t)$ is an AC voltage-source defined as $f(t) = \sin(\omega_0 t)$ where $\omega_0 = 1/\sqrt{LC}$, we are inputting an alternating voltage source oscillating at the *angular resonant angular frequency* of our circuit. (Recall w_0 is called the *angular frequency* of the period-$(2\pi/w_0)$ function $\sin(w_0 t)$; w_0 is measured in radians per time-unit; $w_0/(2\pi)$ is the *frequency* of the period-$(2\pi/w_0)$ function $\sin(w_0 t)$ measured in cycles per time-unit; when the time unit is seconds, the frequency is measured in *hertz*.)

We have

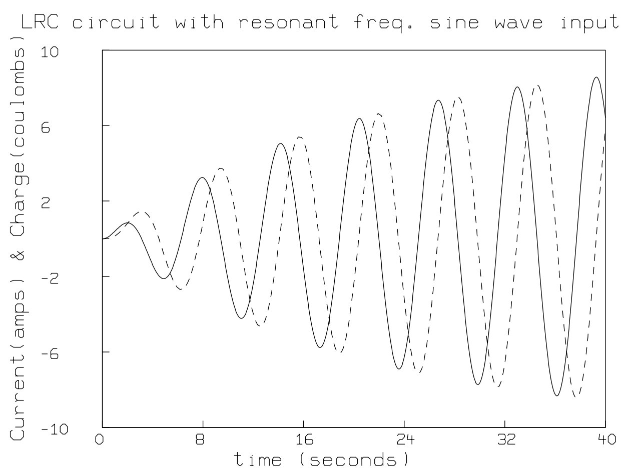
$$L\frac{dI(t)}{dt} + \frac{Q(t)}{C} + RI(t) - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t),$$

$$\text{with } f(t) = \sin(t/\sqrt{LC}), I(0) = 0, \text{ and } Q(0) = 0.$$

The graphs of I and Q for $0 \le t \le 40$ with this driving force function are shown below. The current graph is a solid line, and the charge-difference graph is a dashed line.



In this case, the current and charge-difference in the circuit both *rise* to enter a steady-state oscillation. As the resistance R is reduced, these steady-state oscillations have greater and greater amplitudes. For example, for $R = .1$, we have



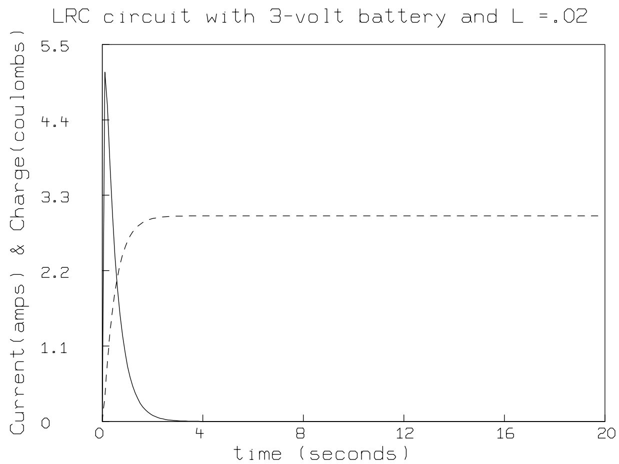
In practice, as R becomes increasingly small, the increasing current will "burn-out" the capacitor, stopping the current.

When the inductor in our circuit has $L = .02$ henries and we return to the case where we have a 3 volt battery DC voltage source, we have

$$.02\frac{dI(t)}{dt} + \frac{Q(t)}{C} + RI(t) - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t),$$

$$\text{with } f(t) = 3, I(0) = 0, \text{ and } Q(0) = 0.$$

The graphs of I and Q for $0 \leq t \leq 20$ are shown below. The current graph is a solid line, and the charge-difference graph is a dashed line.



Note here the current initially rises rapidly to nearly 6 amperes, and then the current and charge-difference in the circuit decays to steady state where the current approaches 0, and the capacitor charge-difference approaches 3 coulombs. With negligible inductance, there is completely-damped oscillation.

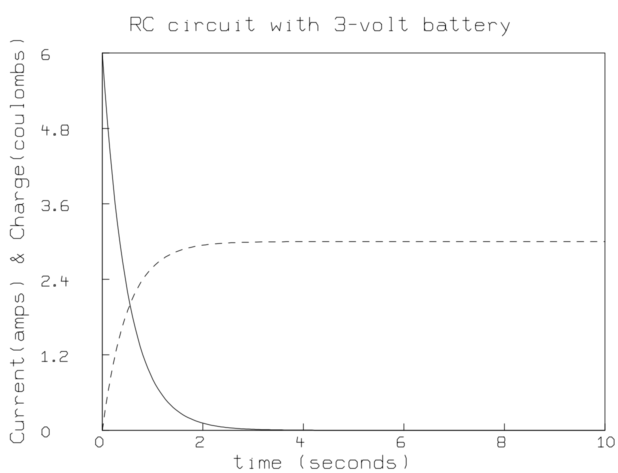
When we completely remove the inductor coil from our circuit, we have $L = 0$, and the equations defining the current and capacitor charge-difference in our circuit become:

$$\frac{Q(t)}{C} + RI(t) - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t), \text{or}$$

$$R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} - f(t) = 0 \text{ and } I(t) = \frac{dQ(t)}{dt},$$

$$\text{with } f(t) = 3 \text{ and } Q(0) = 0.$$

The graphs of I and Q for $0 \leq t \leq 10$ are shown below.



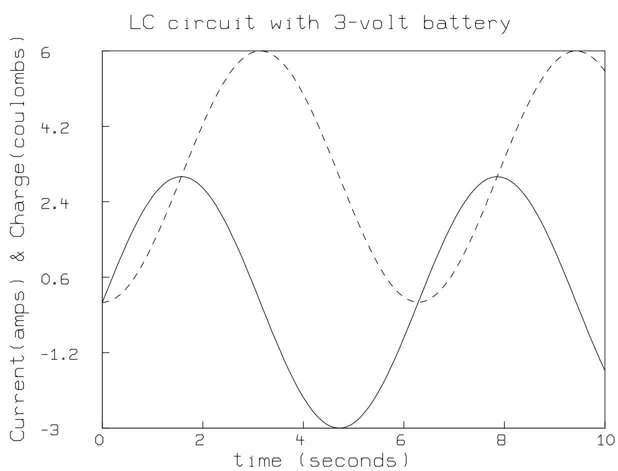
This time there is instantaneous current rise to 6 amperes. (This is not really possible; to be more realistic, we should define the voltage source to have a rapid rise from 0 to 3 volts when it is switched in to more closely approximate the truth.) Again the steady-state current is 0 amperes and the steady-state capacitor charge-difference is 3 coulombs, and with no inductance, there is no oscillation. (Of course no circuit has 0 inductance, and no circuit has 0 resistence either.)

When the inductor in our circuit has $L = 1$ henry and the capacitance is 1 farad, but we have no resistor, we have

$$L\frac{dI(t)}{dt} + \frac{Q(t)}{C} - f(t) = 0 \text{ and } \frac{dQ(t)}{dt} = I(t),$$

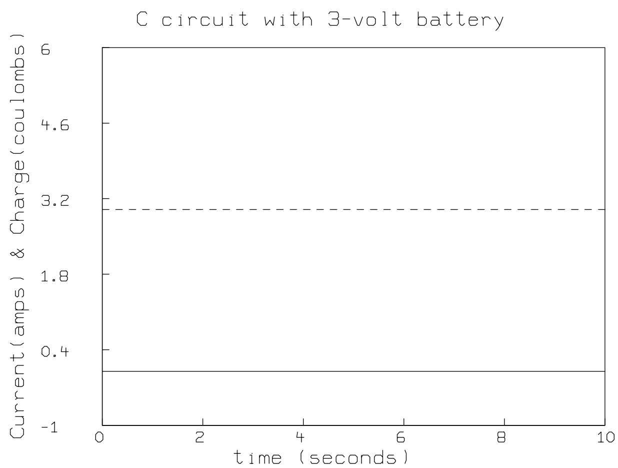
$$\text{with } f(t) = 3, I(0) = 0, \text{ and } Q(0) = 0.$$

The current I and charge-difference Q is shown below. With no resistance, we have no damping, no energy is lost, and our circuit has a pure sine-wave current and capacitor charge-difference out-of-phase by $\pi/2$ radians. The oscillation is due to the opposition of the 1 henry induction coil.



When, in addition to no resistor, there is no inductor coil, we have $Q(t) = Cf(t)$ and $I(t) = C\frac{df(t)}{dt}$ with $f(t) = 3$.

With $C = 1$ farad, and $R = .5$ ohms, the current and capacitor charge-difference in our pure capacitor circuit are shown below.



In this case, our model exhibits a flaw present for all our models; we have a charge difference of 3 coulombs instantly appearing in the capacitor and 0 current when $f(0) \neq 0$, whereas the finite speed-of-light (and charge) makes this impossible. We are instantaneously in steady state. Again, however, in reality no circuit has zero resistence, so this singularity does not practically arise.

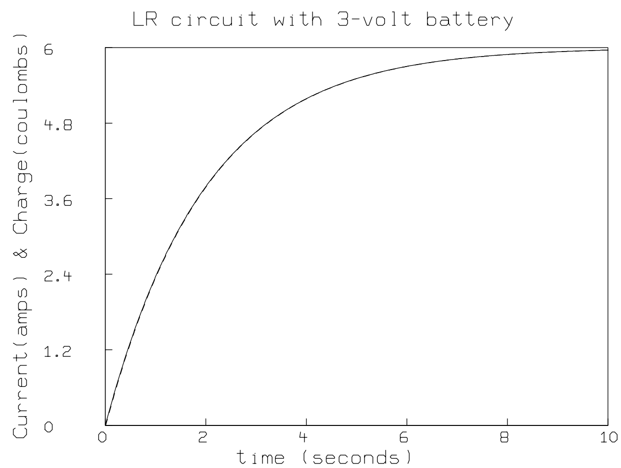
When we "reduce" the capacitor in our circuit to "nothing" to remove the capacitor in our circuit, the charge-difference Q across the capacitor necessarily goes to zero as we imagine the capacitor plates shrinking and approaching one-another to "morph" into a tiny segment of wire. And the capacitance constant diminishes also, but not as fast as the charge-difference Q; thus the voltage drop term $Q(t)/C$ is zero in the limit.

The quantity $Q(t)$ has no meaning in a circuit with no capacitor, but there is still a charge being moved as a current flows in the circuit. Since 1 coulomb of charge is the amount of positive charge moved by a 1 ampere current in 1 second, we may define the *amount of charge* in our circuit at time t (which is identical at every point in the circuit and positive when there is a non-zero current) as the value $Q_a(t) := \int_{t \le \tau \le t+1} |I(\tau)| d\tau$. More properly, we should define $Q_a(t) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t \le \tau \le t+\varepsilon} |I(\tau)| d\tau$. This means the "amount of charge" being moved in our circuit at time t is proportional, with the proportionality constant 1, to the absolute value of the current in our circuit at time t!

With no capacitor, we have the following model.

$$L\frac{dI(t)}{dt} - f(t) - RI(t) = 0 \text{ with I(0) =0, and } Q_a(t) = \textstyle\int_{t \le \tau \le t+1} |I(\tau)| d\tau.$$

Note the graph of Q_a shown below tracks I just as it should.

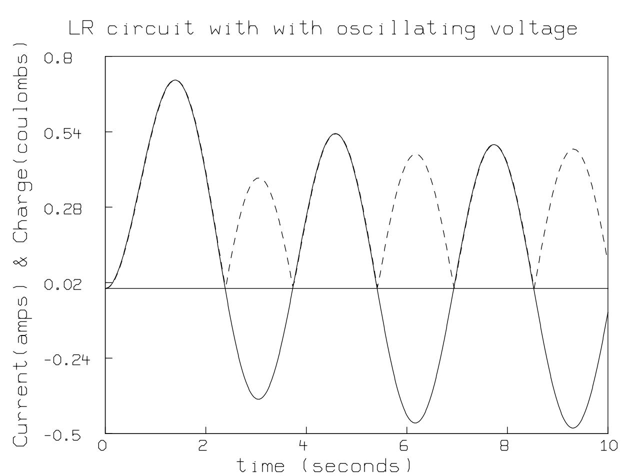


When we impose an oscillating voltage source, $f(t) = sin(2t)$, we have the resulting graph of Q_a shown below tracking $|I|$ just as it should. (If the resonant frequency of our circuit and/or the frequency of our voltage source were smaller, we would need to integrate $|I|$ over a smaller time interval and scale by the reciprocal of the length of the interval of integration to have Q_a computed to track $|I|$.)

Our model and the solution graph is:

$$L\frac{dI(t)}{dt} - f(t) - RI(t) = 0 \text{ and } Q_a(t) = \int_{t \le \tau \le t+1} |I(\tau)| d\tau$$

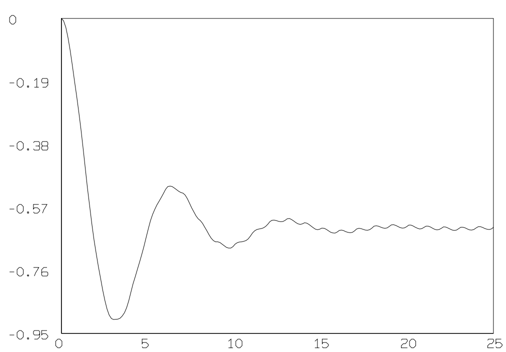
$$\text{with } f(t) = \sin(2t) \text{ and } I(0) = 0.$$



Now let us consider a more dynamic example. Take the initial conditions to be $I(0) = 0$ and $dI(0)/dt = 0$, and define $f(t) = \exp(-(t \bmod 1))$. Fix

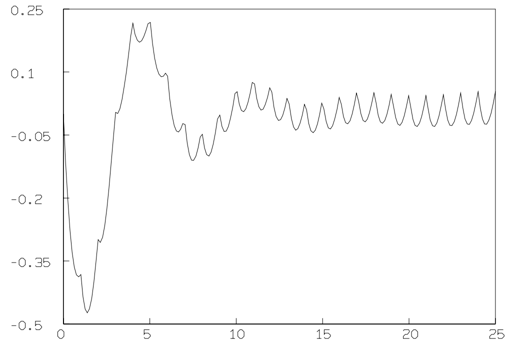
$L = 1$, $C = 1$, and $R = .5$. Now we may solve the differential equation defining the current flow function $I(t)$ and produce a graph of this function as shown below.

```
*function i''t(t)=(f't(t)-r*i't -i/c)/l
*initial i(0) = 0
*initial i't(0) = 0
*function f(t)=exp(-mod(t,1))
*l=1; c=1; r=.5
*m = integrate(i't,i''t,0:25!200)
*type odestr
   odestr = t i't i't't i i't
*draw m col (1,4)
*view
```



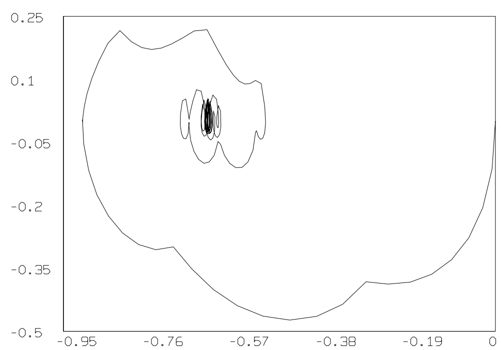
The value of the string variable ODESTR tells us what functions are numerically tabulated in the successive columns of M. The graph of the rate-of-change of the current function $I't(t)$ can also be plotted from the data in M.

```
*delete w
*draw m col 1:2
*view
```



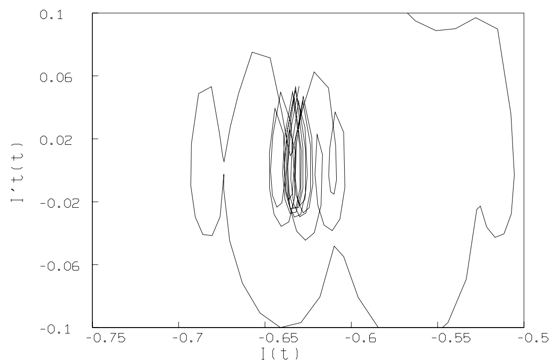
We can display the phase diagram graph for $I(t)$ by plotting $I't(t)$ vs. $I(t)$ as follows

```
*delete w
*draw m col (4,2)
*view
```



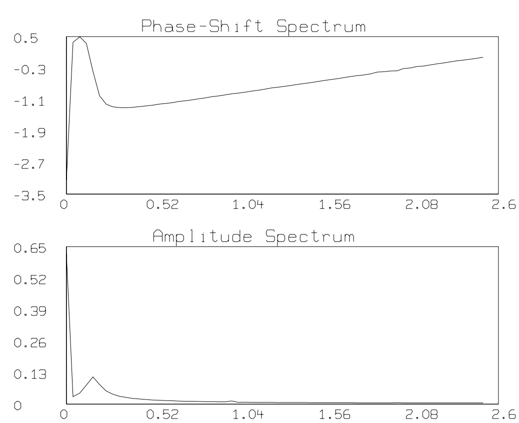
We may "zoom-in" to see the neighborhood of the limit cycle more clearly as follows.

```
*WINDOW -.75 TO -.5, -.1 TO .1
*VIEW
```



We can use the particular MLAB Fourier transform operator **realdft** to compute the amplitude and phase-shift spectra of the current function $I(t)$ tabulated in `M COL (1,4)`.

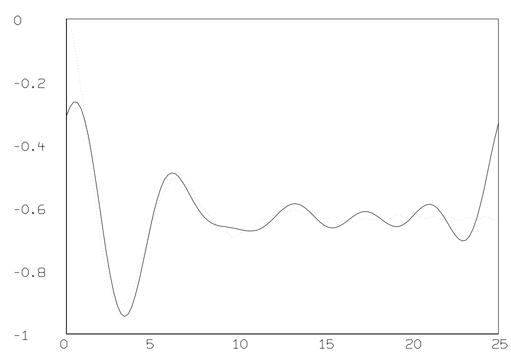
```
*d=realdft(m col(1,4))
*delete w
*draw d col 1:2
*frame 0 to 1, 0 to .5
*top title "Amplitude Spectrum" size .2 inches
*w1=w

*draw d col(1,3)
*frame 0 to 1, .5 to 1
*top title "Phase-Shift Spectrum" size .2 inches
*view
```



Ignoring the large DC value at frequency zero, we see that the maximum amplitude occurs at about .2 Hertz; this is the resonant frequency of the circuit. To see the amplitude spectrum at higher "resolution", we should subtract the DC-value from our signal $I(t)$ and compute the amplitude spectrum of this shifted signal whose mean value is now zero.

The Fourier transform of $I(t)$ contains the information to construct $I(t)$ as a periodic function via its Fourier series. If the Fourier series is truncated, the resulting sum is a filtered form of $I(t)$ omitting the high-frequency components corresponding to the truncated terms. Below we show a graph of the Fourier series of $I(t)$ truncated to 7 terms. Note that Gibbs' phenomenon is exhibited, showing non-uniform convergence to the mid-point of the discontinuities occuring at the points between successive periods of $I(t)$.

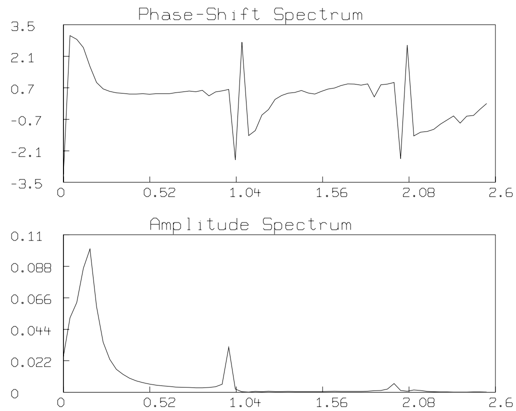
```
*fct s(t)=sum(i,1,n, d(i,2)*cos(2*pi*d(i,1)*t + d(i,3)) )
*n=7
*q=points(s,0:25!120)
*delete w,w1
*draw m col 1:2 lt dotted
*draw q
*view
```



We can also use the MLAB Fourier transform operator to compute the amplitude and phase-shift spectra of the current rate-of-change function $I't(t)$ tabulated in `M COL (1,2)`. As above, this amplitude spectrum shows the resonant frequency of the circuit to be about .2 Hertz. The peak at 1 Hertz corresponds to the frequency of oscillation of the forcing function $f(t)$.

```
*d=realdft(m col 1:2)
*delete w,w1
*draw d col 1:2
*frame 0 to 1, 0 to .5
*top title "Amplitude Spectrum" size .2 inches
*w1=w

*draw d col(1,3)
*frame 0 to 1, .5 to 1
```

```
*top title "Phase-Shift Spectrum" size .2 inches
*view
```



Just as before, the Fourier transform of $I't(t)$ contains the information to construct $I't(t)$ via its Fourier series. If the Fourier series is truncated, the resulting sum is a filtered form of $I't(t)$ omitting the high-frequency components corresponding to the truncated terms. Below we show a graph of the Fourier series of $I't(t)$ truncated to 7 terms, superimposed on a graph of $I't(t)$ plotted as a dotted line.

```
*fct s(t)=sum(i,1,n, d(i,2)*cos(2*pi*d(i,1)*t + d(i,3)) )
*n=7
*q=points(s,0:25!120)
*delete w,w1
*draw m col 1:2 lt dotted
*draw q
*view
```