

# Buoyancy-Driven Boundary-Layer Flow over a Vertical Plate

Gary D. Knott  
Civilized Software, Inc.  
12109 Heritage Park Circle  
Silver Spring, MD 20906 USA  
Tel. (301) 962-3711  
email: csi@civilized.com

In the area of heat transfer and fluid flow, it commonly occurs that the fluid adjacent to a vertical heated plate rises due to the buoyancy force corresponding to the difference in pressure below and above the heated fluid in the vicinity of the vertical plate. This pressure difference is due to the gravity-induced density-gradient in the fluid, taking into account the lower density of the heated fluid; in the absence of gravity, a hot ball of fluid has no tendency to move.

Let us place the origin at the bottom of the vertical plate, with the plate extending along the vertical  $x$ -axis of a left-handed coordinate system. Generally, the heated fluid will flow such that the streamwise flow velocity component in the vertical  $x$ -direction is much greater than the transverse velocity in the horizontal  $y$ -direction (since the  $y$ -direction motion is primarily due to the stream deflection caused by the boundary layer of fluid accreted along the plate.) Also, the  $y$ -direction gradient,  $\partial\phi/\partial y$ , of a field variable  $\phi$  is much greater than the  $x$ -direction counterpart. As a result, most of the flow activity takes place within a thin region adjacent to the plate. The velocity of the heated fluid is small immediately adjacent to the heated plate due to the “friction” associated with the thermal noise in the heated fluid; thus we have non-uniform-velocity laminar flow about the heated plate. As the horizontal distance from the plate increases, the induced velocity rises; and as the horizontal distance increases still more, the induced velocity diminishes due to the increasingly-small pressure difference in excess of the force of gravity.

Nondimensional continuum partial-differential equations governing the boundary-layer flow driven by the buoyancy force mentioned above can be written as

mass conservation (continuity):

$$\partial u / \partial x + \partial v / \partial y = 0,$$

*x*-direction momentum conservation:

$$u\partial u/\partial x + v\partial u/\partial y = (\partial^2 u/\partial y^2)/re - \partial p/\partial x + g_x(1/\rho - 1),$$

*y*-direction momentum conservation:

$$u\partial v/\partial x + v\partial v/\partial y = (\partial^2 v/\partial y^2)/re - \partial p/\partial y + g_y(1/\rho - 1),$$

energy conservation:

$$u\partial T/\partial x + v\partial T/\partial y = \partial^2 T/\partial y^2/pe.$$

The relevant quantities in these equations are defined below.

$u(x, y)$  = dimensionless *x*-direction flow velocity at the point  $(x, y)$ .

$v(x, y)$  = dimensionless *y*-direction flow velocity at the point  $(x, y)$ .

$\rho(x, y)$  = dimensionless density at the point  $(x, y)$ .

$u_0$  = reference flow-velocity value used for dimensionless conversion.

$\rho_0$  = reference density value far from the plate.

$T_0^*$  = reference temperature value far from the plate.

$g_x^*$  = acceleration due to gravity in the vertical *x*-direction.

$g_y^*$  = acceleration due to gravity in the horizontal *y*-direction ( $= 0$ ).

$g_x = g_x^*L/u_0^2$  = dimensionless *x*-direction gravity acceleration.

$g_y = g_y^*L/u_0^2$  = dimensionless *y*-direction gravity acceleration.

$p^*(x, y)$  = the pressure in the fluid at the point  $(x, y)$ .

$p(x, y)$  = dimensionless pressure,  $(p^*(x, y) + \rho_0 g_x L y) / (\rho(x, y) u_0^2)$  at the point  $(x, y)$ .

$\alpha$  = thermal diffusivity.

$\nu$  = kinematic viscosity.

$pe$  = Peclet number,  $u_0 L / \alpha$ .

$re$  = Reynolds number,  $u_0 L / \nu$ .

$T^*(x, y)$  = the temperature in the fluid at the point  $(x, y)$ .

$T(x, y) = (T^*(x, y) - T_0^*) / (T^*(x, 0) - T_0^*)$ .

It is possible and sometimes convenient to transform these partial differential equations into two ordinary differential equations [1].

Let  $s := y(gr/(4x))^{1/4}$  where  $gr$  is the Grashof number  $g_x^* L (T^*(0, 0) - T_0^*) / (u_0^2 T_0^*)$ . Now define  $f(s) := \psi(x, y) / \psi_0(x)$ , where the streamfunction  $\psi(x, y)$  is defined to be  $\int_0^y u(x, \delta) d\delta$  and  $\psi_0(x) := 4\nu(gr/4)^{1/4} x^{3/4}$ . The function  $f$  is a dimensionless streamfunction whose derivative  $f'$  is the *x*-velocity of the fluid in arbitrary units at each point  $(x, y)$  in the vicinity of the heated plate that satisfies  $s = y(gr/(4x))^{1/4}$ . Also define  $h(s) := T(1, s/(gr/4)^{1/4})$ . The dimensionless temperature defined by  $h$  is the constant value  $h(s)$  along

the curve defined by  $y(gr/(4x))^{1/4} = s$ , where the  $x$ -velocity is similarly constant.

With these definitions, and knowing that  $u = \partial\psi/\partial y$  and  $v = -\partial\psi/\partial x$ , we can follow Ostrach [1] to obtain the following system of differential equations. The symbol  $pr$  denotes the Prandtl number,  $\nu/\alpha$ .

$$f''' + 3ff'' - 2(f')^2 + h = 0,$$

$$h'' + 3prfh' = 0,$$

subject to  $f(0) = f'(0) = 0$ ,  $h(0) = 1$ , and  $f'(\infty) = 0$  and  $h(\infty) = 0$ .

This is a boundary-value problem with two boundary-value conditions corresponding to the two unknown initial conditions:  $f''(0) = v_1$  and  $h'(0) = v_2$ . For practical computational purposes,  $f'$  may be taken to be nearly zero at a large finite horizontal distance from the plate; we shall use the finite boundary conditions  $f'(10) = 0$  and  $h(10) = 0$  in place of the infinite boundary conditions given above.

The MLAB mathematical modeling system [2] may be employed to solve this double-shooting problem. The required input, and the corresponding results are shown below. We have constructed an MLAB script-file of MLAB commands entitled *hotair.do*, and executed that script to obtain the results below. (The commands in the script file are echoed in the log-file listing displayed below.)

```
MLAB: Mathematical Modeling System, Revision: January 25, 1996
Copyright: Civilized Software, Inc. (301)652-4714
```

Fri Mar 29 14:03:42 1996

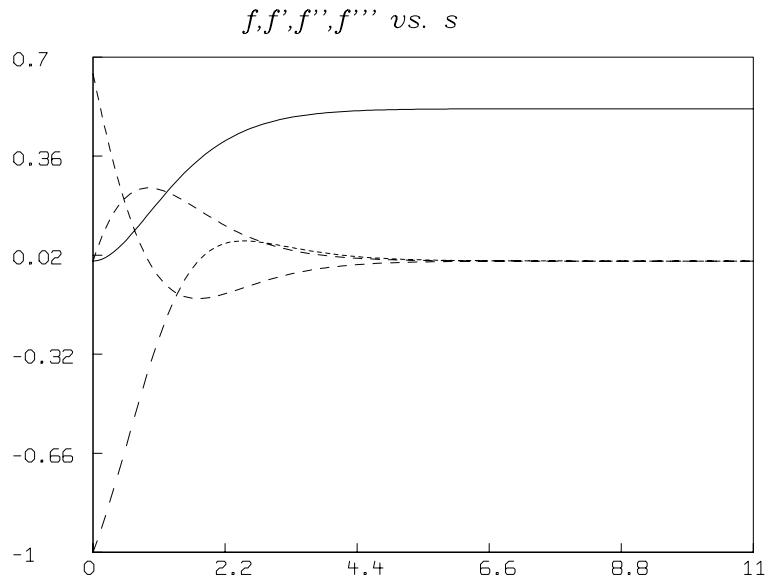
'\*' is the command prompt

```
* do "hotair.do"
*
* fct f'''s(s)=2*(f's)^2-h-3*f*f''s
* fct h''s(s)=-3*pr*f*h's
*
* init f(0)=0
* init f's(0)=0
* init f''s(0)=vf
*
* init h(0)=1
```

```

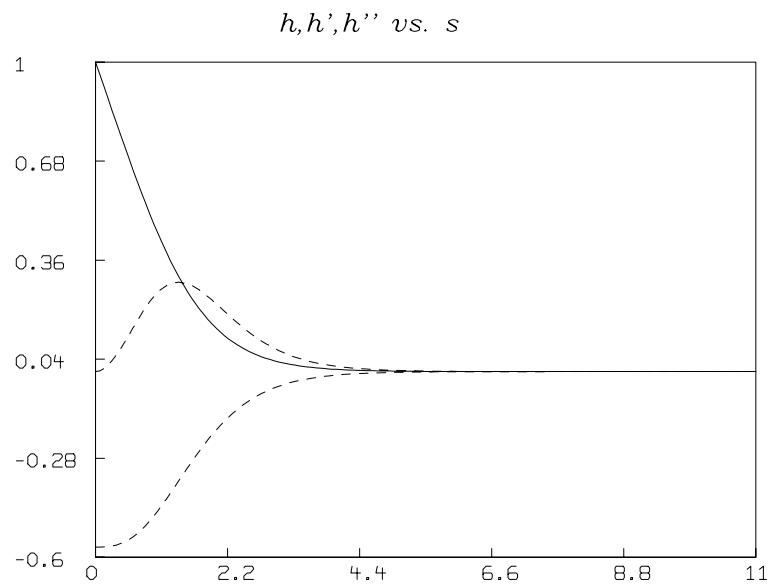
* init h's(0)=vh
*
* pr=1; vf=.5; vh=-1
*
* d=10&'0
*
* constraints q={0<vf,vf<1,vh>-1,vh<0}
* fit(vf,vh), h to d, f's to d, constraints q
final parameter values
      value          error          dependency    parameter
      0.6421470108   3.210436076e-14   0.8268185934    VF
      -0.5671057549  1.688437213e-14   0.8268185934    VH
12 iterations
CONVERGED
best weighted sum of squares = 4.232665e-27
weighted root mean square error = 6.505893e-14
weighted deviation fraction = 1.797693e+308
R squared = 1.000000e+00
no active constraints
*
* m=integrate(f'''s,h's,0:11!160)
* odestr
      ODESTR = S  F'S'S  F'S'S'S  F'S  F'S'S  F  F'S  H'S  H'S'S  H  H'S
*
* draw m col (1,6)
* draw m col (1,4) color red lt dashed
* draw m col (1,2)  lt (.01,004,.01,0,0,0,-1) color green
* draw m col (1,3) color blue lt alternate
* top title "f,f'',f''',f'''''' vs. s" font 17
* view

```



Note that  $f$  is drawn as a solid line, while the derivative functions  $f'$ ,  $f''$ , and  $f'''$  are drawn with dashed lines. It is easy to see which is  $f'$  by looking at the shape of  $f$  in the graph.

```
* delete w
* draw m col (1,10)
* draw m col (1,8) color red lt dashed
* draw m col (1,9) lt (.01,004,.01,0,0,0,-1) color green
* top title "h,h',h''',h'''' vs. s" font 17
* view
```



\* exit

1. S. Ostrach, An analysis of laminar free convection flow and heat transfer about a flat parallel to the direction of the generating body force, NACA, Report 1111, 1953.
2. Civilized Software home-page: <http://www.civilized.com>